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that are sums of two primes

著者	Mikawa Hiroshi
journal or publication title	Tsukuba journal of mathematics
volume	17
number	2
page range	443-453
year	1993-12
URL	http://hdl.handle.net/2241/7259

ON THE INTERVALS BETWEEN CONSECUTIVE NUMBERS THAT ARE SUMS OF TWO PRIMES

By

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1. Introduction.

It is the well known conjecture of H. Cramér that

$$p_{n+1} - p_n \ll (\log p_n)^2$$

where p_n is the n -th prime. In 1940 P. Erdős proposed the problem to estimate the sum

$$\sum_{p_n \leq x} (p_{n+1} - p_n)^2,$$

and A. Selberg showed that it is

$$\ll x(\log x)^3$$

under the Riemann hypothesis. This problem has been stimulating the several authors, vide [2, 3, 10, 11, 13].

Let (g_n) denote in ascending order even integers that are representable as the sum of two primes. The Goldbach conjecture is then interpreted as that

$$g_{n+1} - g_n = 2$$

for all n . In 1952 Ju. V. Linnik [7] proved, on assuming the Riemann hypothesis, that

$$g_{n+1} - g_n \ll (\log g_n)^{3+\varepsilon}$$

for any $\varepsilon > 0$ and all n . Also see [1]. In this paper we shall estimate the third moment of it.

THEOREM.

$$\sum_{g_n \leq x} (g_{n+1} - g_n)^3 \ll x(\log x)^{300}.$$

COROLLARY. For $0 \leq r < 3$, we have

$$\sum_{g_n \leq x} (g_{n+1} - g_n)^r = (2^{r-1} + o(1))x.$$

Our assertion should be compared with the known results in Goldbach's problem. Let $E(x)$ be the number of even integers not exceeding x that may not be expressed as a sum of two primes, and $D(x)$ be the maximum of $(g_{n+1} - g_n)$ for $g_n \leq x$. It was proved by H. L. Montgomery and R. C. Vaughan [9] that

$$E(x) \ll x^{1-\delta}$$

with some $\delta > 0$. As for $D(x)$, the argument in [9] runs as follows. Suppose that one knows the equi-distribution of primes in intervals $[x, x+x^\theta]$ for almost all x , and in $[x, x+x^\theta]$ for all x . Then,

$$(1.1) \quad D(x) \ll x^{\theta\theta}.$$

By an elementary consideration, see section 3, we find

$$\sum_{g_n \leq x} (g_{n+1} - g_n)^2 = 2x + O(D(x)E(x)).$$

It seems that no unconditional result leads $\theta\theta \leq \delta$.

Our argument is based upon Linnik's method [6, 7] and D. Wolke's trick [13]. The limitation of our estimate comes from A. E. Ingham's bound [4] for zeros of the Riemann zeta-function.

I would like to thank Professor Uchiyama and Dr. Kawada for encouragement and valuable comment.

2. Notation and Lemmas.

We use the standard notation in number theory. ρ stands for the non-trivial zeros of the Riemann zeta-function. For $1/2 \leq \sigma \leq 1$ and $T > 0$, $N(\sigma, T)$ denotes the number of ρ such that $\sigma \leq \text{Re}(\rho)$ and $|\text{Im}(\rho)| \leq T$.

LEMMA 1. *Uniformly for $x, T \geq 3$, we have*

$$\sum_{|\text{Im}(\rho)| \leq T} x^\rho = -\frac{T}{\pi} A'(x) + O(x(\log xT)^2)$$

where $A'(x)$ is equal to the von Mangoldt function if x is an integer, and $A'(x) = 0$ otherwise.

This is a formula of E. Landau [5]. Though his estimate is not uniform for x , it is easy to alter the proof of [5] to be suitable for our aim. The following Lemma 2 is due to Ingham [4] and Montgomery [8, Theorem 1]. Lemma 3 follows from [8, Theorem 2].

LEMMA 2. For $T > 2$, we have

$$N(\sigma, T) \ll T^{\lambda(\sigma)(1-\sigma)} (\log T)^{13}$$

where

$$\lambda(\sigma) = \begin{cases} \frac{3}{2-\sigma} & \text{if } 1/2 \leq \sigma \leq 4/5 \\ \frac{2}{\sigma} & \text{if } 4/5 \leq \sigma \leq 1. \end{cases}$$

LEMMA 3. If $9/10 \leq \sigma \leq 1$, then

$$N(\sigma, T) \ll T^{(2-c)(1-\sigma)} (\log T)^{13}$$

where c is a positive absolute constant.

In sections 3 and 4 we use the convention $L = \log X$. For a real x , write $e(x) = e^{2\pi i x}$. $*$ and \wedge mean that $f * g(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy$ and $\hat{f}(x) = \int_{-\infty}^{+\infty} f(y) \cdot e(-xy)dy$, respectively. The implied constants in O and \ll are absolute, except for the proof of Corollary.

3. Reduction of the problem

In this section we first reduce the proof of Theorem to that of Lemma 4 below. Lemma 4 will be verified in section 5. Next we derive Corollary from Theorem. Put $d_n = g_{n+1} - g_n$, for simplicity.

PROOF OF THEOREM. It is sufficient to prove

$$F(x) = \sum_{x' < g_n \leq x} d_n^3 \ll x(\log x)^{300}$$

for all large x and $x' = (5/7)x$. We have

$$F(x) \ll \sum_{\substack{x' < g_n \leq x \\ d_n \leq (\log x)^{150}}} d_n^3 + (\log x) \sup_{\delta > (\log x)^{150}} \sum_{\substack{x' < g_n \leq x \\ \delta < d_n \leq 2\delta}} d_n^3.$$

Because of (1.1), $\delta \leq D(x) < x^{1/6}$. Put

$$\Gamma(x, \delta) = \{g_n : x' < g_n \leq x, \delta < d_n \leq 2\delta, g_{n+1} \leq x\}.$$

Then,

$$\begin{aligned} (3.1) \quad F(x) &\ll (\log x)^{300} \sum_{g_n \leq x} d_n + (\log x) \sup_{(\log x)^{150} < \delta < x^{1/6}} \left(\sum_{g_n \in \Gamma(x, \delta)} d_n^3 + \delta^2 D(x) \right) \\ &\ll x(\log x)^{300} + (\log x) \sup_{(\log x)^{150} < \delta < x^{1/6}} \delta^2 \sum_{g_n \in \Gamma(x, \delta)} d_n. \end{aligned}$$

Here we state our main lemma.

LEMMA 4. Let X be a large parameter,

$$(5/2)X \leq x \leq (7/2)X \quad \text{and} \quad (1/2)(\log X)^{150} < \Delta < X^{1/6}.$$

There exists a function $R(x, \Delta)$ such that

$$(3.2) \quad \int_{(5/2)X}^{(7/2)X} |R(x, \Delta)|^2 dx \ll X^3 (\log X)^{299}$$

and

$$(3.3) \quad \sum_{\substack{X < m, n \leq 2X \\ x - \Delta < m + n \leq x}} A(m)A(n) = \Delta(X - |x - 3X|) + O(\Delta X (\log X)^{-4}) + R(x, \Delta),$$

uniformly for X, x and Δ .

Now, if $t \in [(g_n + g_{n+1})/2, g_{n+1})$ for $g_n \in \Gamma(x, \delta)$ then

$$t - \frac{\delta}{2} > \frac{g_n + g_{n+1}}{2} - \frac{d_n}{2} = g_n.$$

Namely the interval $(t - \delta/2, t]$ contains no sum of two primes. By (3.3) in Lemma 4 with $(7/2)X = x$ and $2\Delta = \delta$, we therefore have

$$R(t, \delta/2) = -\frac{\delta}{2} \left(\frac{2}{7}x - \left| t - \frac{6}{7}x \right| \right) + O(\delta x (\log x)^{-4})$$

for all $t \in [(g_n + g_{n+1})/2, g_{n+1})$ with $g_n \in \Gamma(x, \delta)$. Since these intervals are mutually disjoint, we have

$$\begin{aligned} \sum_{g_n \in \Gamma(x, \delta)} \left(g_{n+1} - \frac{g_n + g_{n+1}}{2} \right) (\delta x)^2 &\ll \sum_{g_n \in \Gamma(x, \delta)} \int_{(g_n + g_{n+1})/2}^{g_{n+1}} |R(t, \delta/2)|^2 dt \\ &\leq \int_{x'}^x |R(t, \delta/2)|^2 dt. \end{aligned}$$

Hence (3.2) in Lemma 4 yields that

$$\delta^2 \sum_{g_n \in \Gamma(x, \delta)} d_n \ll x (\log x)^{299}$$

uniformly for $\delta, (\log x)^{150} < \delta < x^{1/6}$. Combining this with (3.1) we obtain

$$F(x) \ll x (\log x)^{300},$$

as required.

PROOF OF COROLLARY. With the notation in section 1, we easily see that

$$\sum_{g_n \leq x} d_n = x + O(D(x)),$$

and

$$\sum_{g_n \leq x} 1 = \frac{1}{2}x + O(1) - E(x).$$

By subtraction, we have

$$(3.4) \quad \sum_{\substack{g_n \leq x \\ d_n > \frac{x}{2}}} d_n \ll D(x) + E(x) \ll E(x),$$

or

$$(3.5) \quad \sum_{\substack{g_n \leq x \\ d_n > \frac{x}{2}}} 1 = \frac{1}{2} x + O(E(x)).$$

Now, if $0 \leq \gamma \leq 1$ then

$$\sum_{g_n \leq x} d_n^r = 2^r \sum_{\substack{g_n \leq x \\ d_n = 2}} 1 + O\left(\sum_{\substack{g_n \leq x \\ d_n > \frac{x}{2}}} d_n\right) = 2^{r-1} x + O(E(x))$$

by (3.4) and (3.5). It is known [12; Kap. VI. Satz 7.1] that

$$(3.6) \quad E(x) \ll x(\log x)^{-A}$$

for any $A > 0$. Hence we get Corollary in case $0 \leq \gamma \leq 1$.

Suppose $1 < \gamma < 3$. Let D be a positive constant, which will be specified later. Then,

$$\begin{aligned} \sum_{g_n \leq x} d_n^r &= \sum_{d_n=2} + \sum_{2 < d_n \leq (\log x)^D} + \sum_{d_n > (\log x)^D} \\ &= 2^{r-1} x + O(E(x)) + O\left((\log x)^{(\gamma-1)D} \sum_{\substack{g_n \leq x \\ d_n > \frac{x}{2}}} d_n\right) + O\left((\log x)^{-(3-\gamma)D} \sum_{g_n \leq x} d_n^3\right) \\ &= 2^{r-1} x + O(E(x)(\log x)^{(\gamma-1)D}) + x(\log x)^{300-(3-\gamma)D} \end{aligned}$$

because of (3.4), (3.5) and Theorem. On taking $D = 301/(3-\gamma)$ we get, by (3.6), that

$$\sum_{g_n \leq x} d_n^r = 2^{r-1} x + O(x(\log x)^{-1}),$$

as required.

4. Proof of Lemma 4, preliminaries.

We begin with modifying the explicit formula:

$$(4.1) \quad \sum_{n \leq x} \Lambda(n) = x - \sum_{|\operatorname{Im}(\rho)| \leq T} \frac{x^\rho}{\rho} + O\left(\left(1 + \frac{x}{T}\right)(\log x T)^2\right)$$

uniformly for $x, T \geq 3$. For $T \geq 3$, define

$$(4.2) \quad q_n = q_n(T) = \int_{n-1/2}^{n+1/2} \sum_{|\operatorname{Im}(\rho)| \leq T} y^{\rho-1} dy$$

if $n \leq 5$, and $q_n = 0$ otherwise. Moreover we determine $r_n = r_n(T)$ by the relation

$$(4.3) \quad \Lambda(n) = 1 - q_n - r_n.$$

Lemma 1 then gives

$$(4.4) \quad q_n, r_n \ll (\log nT)^2.$$

For large x , it follows from the prime number theorem, (4.1) and (4.2) that

$$(4.5) \quad \sum_{n \leq x} q_n = \sum_{|\operatorname{Im}(\rho)| \leq T} \left(\frac{(\lfloor x \rfloor + 1/2)^\rho}{\rho} + O\left(\frac{1}{|\rho|}\right) \right) \\ \ll x \exp(-(\log x)^{1/2}) + \left(1 + \frac{x}{T}\right) (\log xT)^2.$$

Similarly,

$$(4.6) \quad \sum_{n \leq x} r_n \ll \left(1 + \frac{x}{T}\right) (\log xT)^2$$

by (4.1), (4.2) and (4.3).

Now, on choosing

$$T = \frac{X}{\Delta} L^8,$$

we consider the sum in question:

$$G = \sum_{\substack{X < m, n \leq 2X \\ x - \Delta < m + n \leq x}} A(m)A(n).$$

By (4.3),

$$A(m)A(n) = 1 + q_m q_n - (q_m + q_n) - r_m A(n) - A(m) r_n - r_m r_n.$$

Accordingly,

$$(4.7) \quad G = G_1 + G_2 - 2G_3 - 2G_4 - G_5, \quad \text{say.}$$

$$(4.8) \quad G_1 = \sum_{\substack{X < m, n \leq 2X \\ x - \Delta < m + n \leq x}} 1 \\ = \sum_{\substack{X < m \leq 2X \\ x - 2X < m < x - \Delta - X}} \# \{n : x - m - \Delta < n \leq x - m\} + O(\Delta^2) \\ = \Delta \sum_{\substack{X < m \leq 2X \\ x - 2X < m \leq x - X}} 1 + O(X) \\ = \Delta(X - |x - 3X|) + O(X).$$

On writing

$$(4.9) \quad Z(y) = Z(y, T) = \sum_{|\operatorname{Im}(\rho)| \leq T} y^{\rho-1},$$

we have

$$\begin{aligned}
 G_2 &= \sum_{\substack{X < m, n \leq 2X \\ x - \Delta < m + n \leq x}} \int_{m-1/2}^{m+1/2} \int_{n-1/2}^{n+1/2} Z(u)Z(v) du dv \\
 &= \iint_D Z(u)Z(v) du dv, \quad \text{say.}
 \end{aligned}$$

We replace the domain D by

$$(4.10) \quad D = D(X, x, \Delta) = \{(u, v) \in [X, 2X]^2 : x - \Delta \leq u + v \leq x\}.$$

The resulting error is

$$(4.11) \quad \ll \iint_{(D \cup D) \setminus (D \cap D)} |Z(u)Z(v)| du dv \ll XL^4,$$

because of Lemma 1.

$$\begin{aligned}
 (4.12) \quad G_3 &= \sum_{\substack{X < m, n \leq 2X \\ x - \Delta < m + n \leq x}} q_m \\
 &= \sum_{\substack{X < m \leq 2X \\ x - 2X < m \leq x - \Delta - X}} q_m (\Delta + O(1)) + O(\Delta^2 L^2) \\
 &\ll \Delta \sup_{M \leq 2X} \left| \sum_{m \leq M} q_m \right| + XL^2 \\
 &\ll \Delta XL^{-4},
 \end{aligned}$$

by (4.4) and (4.5). Also, (4.4) and (4.6) give that

$$\begin{aligned}
 (4.13) \quad G_4 &= \sum_{\substack{X < m, n \leq 2X \\ x - \Delta < m + n \leq x}} \Lambda(m)r_n \\
 &\ll \sum_{m \leq 2X} \Lambda(m) \sup_{N \leq 2X} \left| \sum_{n \leq N} r_n \right| \\
 &\ll XL^2 \left(1 + \frac{X}{T}\right) L^2 \\
 &\ll \Delta XL^{-4}.
 \end{aligned}$$

Similarly,

$$(4.14) \quad G_5 \ll \Delta XL^{-4}.$$

On summing up the above estimates (4.7)–(4.14) we obtain

$$(4.15) \quad G = \Delta(X - |x - 3X|) + O(\Delta XL^{-4}) + \iint_D Z(u)Z(v) du dv$$

where Z and D are defined by (4.9) and (4.10), respectively.

5. Proof of Lemma 4.

Put

$$N^+(\sigma, T) = T^{\lambda(\sigma)(1-\sigma)} L^{13}$$

where λ is defined in Lemma 2. Since

$$T^{\lambda(1/2)} = T^{\lambda(1)} = \left(\frac{X}{\Delta} L^8\right)^2 < X^2 L^{-280} < \left(\frac{X}{\Delta} L^8\right)^{12/5} = T^{\lambda(3/4)} < T^{\lambda(4/5)},$$

there exist r and t such that $1/2 < r < 3/4$, $4/5 < t < 1$ and

$$T^{\lambda(r)} = T^{\lambda(t)} = X^2 L^{-280}.$$

Define $s = \min(t, 9/10)$, and $I = [r, s]$. We then see

$$(5.1) \quad T^{\lambda(\sigma)} \leq X^2 L^{-280} \quad \text{for all } \sigma \in [1/2, 9/10] \setminus I,$$

and

$$(5.2) \quad T^{\lambda(\sigma)} \geq X^2 L^{-280} \quad \text{for all } \sigma \in I.$$

Now, we divide the sum $Z(y)$, which is defined by (4.9).

$$(5.3) \quad Z(y) = \sum_{\operatorname{Re}(\rho) \notin I} + \sum_{\operatorname{Re}(\rho) \in I} = z_1(y) + z(y), \quad \text{say.}$$

We first consider z_1 . By a familiar way,

$$\begin{aligned} J &= \int_X^{2X} |z_1(y)|^2 dy \ll L^2 \sum_{\substack{\operatorname{Im}(\rho) \leq T \\ \operatorname{Re}(\rho) \notin I}} X^{2\operatorname{Re}(\rho)-1} \\ &\ll L^2 \sum_{\substack{\operatorname{Im}(\rho) \leq T \\ \operatorname{Re}(\rho) < 1/2}} 1 + L^3 \sup_{1/2 \leq \sigma \notin I} X^{2\sigma-1} N(\sigma, T). \end{aligned}$$

Here, because of the zero-free region [12; Kap. VIII. Satz 6.2], the above supremum may be taken over $\sigma \leq 1 - \eta(T)$ only, where $\eta(T) = (\log T)^{-4/5}$. Lemmas 2 and 3 yield that

$$\begin{aligned} (5.4) \quad J &\ll L^3 T + L^{16} X \left\{ \sup_{\substack{1/2 \leq \sigma < 9/10 \\ \sigma \notin I}} \left(\frac{T^{\lambda(\sigma)}}{X^2} \right)^{1-\sigma} + \sup_{9/10 \leq \sigma \leq 1-\eta(T)} \left(\frac{T^2}{X^2} \right)^{1-\sigma} T^{-c(1-\sigma)} \right\} \\ &\ll L^{11} X \Delta^{-1} + L^{16} X \{ (X^{-280})^{1/10} + T^{-c\eta(T)} \} \\ &\ll XL^{-12}, \end{aligned}$$

by (5.1).

We turn to the double integral in (4.15). Since

$$Z(u)Z(v) = z(u)z(v) + z_1(u)Z(v) + Z(u)z_1(v) - z_1(u)z_1(v),$$

$$\begin{aligned}
& \iint_D (Z(u)Z(v) - z(u)z(v)) du dv \\
& \ll \iint_{\substack{X \leq u, v \leq 2X \\ x - \Delta \leq u+v \leq x}} |z_1(u)| (|Z(v)| + |z_1(v)|) du dv \\
& \ll L^2 \Delta \int_X^{2X} |z_1(y)| dy + \Delta \int_X^{2X} |z_1(y)|^2 dy \\
& \ll L^2 \Delta (X^2 L^{-12})^{1/2} + \Delta X L^{-12} \\
& \ll \Delta X L^{-4},
\end{aligned}$$

by Lemma 1 and (5.4). Combining this with (4.15) we reach (3.3);

$$G = \Delta(X - |x - 3X|) + O(\Delta X L^{-4}) + R(x, \Delta)$$

where

$$(5.5) \quad R(x, \Delta) = \iint_D z(u)z(v) du dv.$$

It remains to prove (3.2). First we define $z(y) = 0$ if $y \notin [X, 2X]$. Next we split up $z(y)$. Let $z_\sigma(y)$ be the partial sum of $z(y)$ restricted by $\sigma \leq \text{Re}(\rho) < \sigma(1+1/L)$. Then,

$$z(y) = \sum_{\substack{\alpha = r(1+1/L) \\ n \geq 0}} z_\alpha(y).$$

Furthermore let $\chi(x)$ denote the characteristic function of $[0, \Delta]$. Thus we may rewrite (5.5) as

$$\begin{aligned}
R(x, \Delta) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \chi(x-u-v) z(u)z(v) du dv \\
&= \chi * z * z(x).
\end{aligned}$$

Now, by Plancherel's relation, we have

$$\begin{aligned}
(5.6) \quad I &= \int_{(5/2)X}^{(7/2)X} |R(x, \Delta)|^2 dx \leq \int_{-\infty}^{+\infty} |\chi * z * z(x)|^2 dx \\
&= \int_{-\infty}^{+\infty} |\widehat{\chi * z * z}(x)|^2 dx \\
&= \int_{-\infty}^{+\infty} |\hat{\chi}(x)|^2 |\hat{z}(x)|^4 dx.
\end{aligned}$$

Here we see

$$|\hat{\chi}(x)|^2 = \left(\frac{\sin \pi \Delta x}{\pi x} \right)^2,$$

and, on using Hölder's inequality,

$$|\hat{z}(x)|^4 \ll L^3 \sum_{\alpha} |\hat{z}_{\alpha}(x)|^4.$$

Therefore (5.6) becomes

$$(5.7) \quad I \ll L^4 \Delta^2 \sup_{\sigma \in I} \left(\sup_x |\hat{z}_{\sigma}(x)|^2 \right) \int_{-\infty}^{+\infty} |z_{\sigma}(x)|^2 dx,$$

by Plancherel's relation again.

We proceed to estimate the square integral of z_{σ} .

$$(5.8) \quad \int_{-\infty}^{+\infty} |z_{\sigma}(y)|^2 dy = \int_X \left| \sum_{\substack{|\operatorname{Im}(\rho)| \leq T \\ \sigma \leq \operatorname{Re}(\rho) < \sigma(1+1/L)}} y^{\rho-1} \right|^2 dy \\ \ll L^2 X^{2\sigma-1} N(\sigma, T).$$

We turn to \hat{z}_{σ} . The simplest saddle point method [12; Kap IX, Lemma 4.2] leads that

$$\begin{aligned} \hat{z}_{\sigma}(x) &= \sum_{\substack{|\operatorname{Im}(\rho)| \leq T \\ \sigma \leq \operatorname{Re}(\rho) < \sigma(1+1/L)}} \int_X y^{\operatorname{Re}(\rho)-1} \exp(i(\operatorname{Im}(\rho) \log y - 2\pi xy)) dy \\ &\ll L X^{\sigma} + \sum_{\substack{3 \leq |\operatorname{Im}(\rho)| \leq T \\ \sigma \leq \operatorname{Re}(\rho) < \sigma(1+1/L)}} X^{\operatorname{Re}(\rho)} |\operatorname{Im}(\rho)|^{-1/2} \\ &\ll L X^{\sigma} \left(1 + \sup_{3 \leq t \leq T} t^{-1/2} N(\sigma, t) \right). \end{aligned}$$

We now appeal to Lemma 2. Since $\lambda(\sigma)(1-\sigma) \geq 1/2$ if $1/2 \leq \sigma \leq 4/5$ and $\leq 1/2$ if $4/5 \leq \sigma \leq 1$, we have that

$$(5.9) \quad \hat{z}_{\sigma}(x) \ll \begin{cases} L X^{\sigma} T^{-1/2} N^{+}(\sigma, T) & \text{if } 1/2 \leq \sigma \leq 4/5 \\ L^{14} X^{\sigma} & \text{if } 4/5 \leq \sigma \leq 1, \end{cases}$$

uniformly for x .

In conjunction with (5.7), (5.8) and (5.9) we obtain

$$I \ll L^4 \Delta^2 \left(\sup_{\substack{\sigma \in I \\ \sigma \leq 4/5}} L^4 X^{4\sigma-1} T^{-1} N^{+}(\sigma, T)^3 + \sup_{\substack{\sigma \in I \\ \sigma \geq 4/5}} L^{30} X^{4\sigma-1} N^{+}(\sigma, T) \right).$$

Notice that

$$\lambda(\sigma)(1-\sigma) = \begin{cases} 1 - \frac{1}{3} \lambda(\sigma)(2\sigma-1) & \text{if } 1/2 \leq \sigma \leq 4/5 \\ 2 - \lambda(\sigma)(2\sigma-1) & \text{if } 4/5 \leq \sigma \leq 1. \end{cases}$$

Hence, by (5.2), we conclude

$$\begin{aligned} I &\ll L^{47} \Delta^2 \sup_{\sigma \in I} X^{4\sigma-1} T^{2-\lambda(\sigma)(2\sigma-1)} \\ &\ll L^{63} X^3 \sup_{\sigma \in I} \left(\frac{X^2}{T^{\lambda(\sigma)}} \right)^{2\sigma-1} \\ &\ll X^3 L^{287}, \end{aligned}$$

as required.

This completes our proof.

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